International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

Quantum Space-Time and Tetrads

Holger Lyre[1]

Received July 4, 1997

The description of space-time in a quantum-theoretic framework must be considered as a fundamental problem in physics. Most attempts start with an already given classical space-time, then the quantization is done. In contrast to this, the central assumption in this paper is not to start with space-time, but to derive it from some more abstract presuppositions as done in Von Weizsäcker's quantum theory of ur-alternatives. Mathematically, the transition from a manifold with spin structure to a manifold with four real space-time coordinates has to be considered. The suggestion is made that this transition can be well described by using a tetradial formalism which appears to be the most natural connection between ur-spinors and real four-vectors.

1. INTRODUCTION

This paper is a speculative one; it deals with some perspectives in describing space-time from a more fundamental point of view: from an underlying "world" of spinors. Our starting point is the so-called *quantum theory of ur-alternatives* which is based on the physical and philosophical considerations of Von Weizsäcker (1985). The idea is that the primary substratum in the world is given by urs: the simplest objects in quantum theory. Conceptually they represent nothing further than *one bit of potential information* (Lyre, 1995). Thus, they could be called quantum bits (in modern quantum information theory sometimes called "qubits"). It is mathematically trivial that each physical object in quantum theory may be embedded in a tensor product of urs. Let us consider the essential symmetry group of urs, which is

$$U(2) = SU(2) \otimes U(1) \sim \mathbb{S}^3 \times \mathbb{S}^\perp \tag{1}$$

In ur-theory the central assumption is that the group-manifold $\mathbb{S}^3 \times \mathbb{R}^+$, which is associated to the universal symmetry group of urs $U(2)$, has to be looked

[1] Institute of Philosophy, Ruhr-University Bochum. D-44780 Bochum, Germany; e-mail: holger.lyre@rz.ruhr-uni-bochum.de.

0020-7748/98/0100-0393$15.00/0

upon as a model of our global cosmic space and time (we take $\mathbb{R}^+$ instead of $\mathbb{S}^1$ to describe time because of our philosophical motivation: we have to presuppose the difference between past and future which is essential in empirical science).

What is the argument for that astonishing assumption? As far as any empirical object is build up from urs, the symmetry properties of urs have to be the symmetry properties of any empirical objects, but the latter are essentially the space-time symmetry properties. Thus, space and time are not just given as a universal background in physics, but reflect the fundamental structure of ur-spinors. A familiar assumption on deriving space-time from spinors had been given by Penrose (1984) in his twistor theory. Finkelstein (1994) gave a suggestive name to these kinds of programs: "spinorism."

To summarize our starting point:

Basic Assumption. Ur-spinors are to be considered as the fundamental physical entities. The quantum theory of urs leads to the universal symmetry group

$$U(2) = SU(2) \otimes U(1)$$

Since physical objects are built up from urs, the symmetry group of urs (conceived as a homogeneous space of the group itself) gives a model of global space-time

$$\mathbb{S}^3 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3 \times \mathbb{R}^+$$

Conclusion 1. The three-dimensionality of position space and the one-dimensionality of time are derived. The first approximation of global position-space, i.e., the cosmic model $\mathbb{S}^3$, is characterized as a maximal symmetric space which allows a Killing group with six parameters which is given by $SO(4)$. The curvature of our cosmos is $k = 1$. In the static case this is an Einstein cosmos.

2. UR-THEORY IN THE GLOBAL EINSTEIN COSMOS

2.1. Ur-Spinors

According to Conclusion 1, urs are to be considered as nonlocalized functions in the global Einstein cosmos $\mathbb{S}^3$. We choose a special parametrization to represent them. A general element of $U(2)$ can be written as

$$A = Ue^{i\varphi} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} e^{i\varphi}, \qquad a, b \in \mathbb{C}, \quad \varphi \in \mathbb{R} \tag{2}$$

From the unitarity condition

$$A^{+}A = U^{+}U = \mathbb{1}_{2\times 2} \tag{3}$$

we have

$$\det U = a^*a + b^*b = 1 \tag{4}$$

With $a = w + iz$ and $b = y + ix$ this is equivalent to

$$w^2 + x^2 + y^2 + z^2 = 1 \tag{5}$$

i.e., a representation of $\mathbb{S}^3$. Thus, urs are nonlocalized functions on this group manifold. For example, the two columns of (2) represent the ur-spinors u^A and v^A with components

$$u^1 = a, \qquad u^2 = -b^*, \qquad u^1 = b, \qquad v^2 = a^* \tag{6}$$

In spinor space we use the metric

$$(\varepsilon_{AB}) = (\varepsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{7}$$

which acts like

$$u_A = \varepsilon_{AB}u^B, \qquad u^A = \varepsilon^{BA}u_B = u_B\varepsilon^{BA} \tag{8}$$

and therefore the covariant components of (6) are

$$\begin{aligned} u_1 &= u^2 = -b^*, \qquad u_2 = -u^1 = -a, \\ u_1 &= v^2 = a^*, \qquad u_2 - v^1 = -b \end{aligned} \tag{9}$$

Hence the ur-spinors are orthogonal,

$$u_Au^A = u_Au^A = 0 \tag{10}$$

and fulfill the conditions

$$u_Au^A = -u_Au^A = 1 \tag{11}$$

Thus, the ur-spinorial system represents a *dyad.*

2.2. Ur-Tetrads

The main motivation of ur-theory is the foundation of space-time structure from the symmetry of urs. Therefore, we have to look for an appropriate mathematical tool to express this. Because of the equivalence of a spinorial dyad to a tensorial tetrad, such a tool is properly given by the tetradial formalism. A tetrad (vierbein) must be looked upon as a spatial reference frame which is represented by a system of four linear independent 4-vectors

$t_\mu^{(\alpha)}$ numbered by the index put in brackets. The metric tensor of space-time is given by

$$q_{\mu\nu} = g_{(\alpha)(\beta)} t_\mu^{(\alpha)} t_\nu^{(\beta)} \tag{12}$$

whereas $g_{(\alpha)(\beta)}$ obeys the condition

$$g_{(\alpha)(\beta)} g^{(\beta)(\gamma)} = g_{(\alpha)}^{(\gamma)} = \delta_{(\alpha)}^{(\gamma)} \tag{13}$$

and defines the lower tetradial indices

$$t_{(\alpha)\mu} = g_{(\alpha)(\beta)} t_\mu^{(\beta)} \tag{14}$$

A special tetrad is given by using null vectors, i.e., $t_\mu^{(\alpha)}\ t^{\mu(\alpha)} = 0$. It turns out that the relations (10) and (11) are suitable to define four lightlike 4-vectors in the following way:

$$l^\mu = \frac{1}{\sqrt{2}} \sigma^\mu_{\dot{A}B} v^{\dot{A}} u^B, \qquad m^\mu = \frac{1}{\sqrt{2}} \sigma^\mu_{\dot{A}B} v^{\dot{A}} v^B$$
$$l^{*\mu} = \frac{1}{\sqrt{2}} \sigma^\mu_{\dot{A}B} u^{\dot{A}} v^B, \qquad n^\mu = \frac{1}{\sqrt{2}} \sigma^\mu_{\dot{A}B} u^{\dot{A}} u^B \tag{15}$$

whereas the dotted indices denote the complex conjugate spinor components and $\sigma\mu$ are the Pauli-matrices. The vectors (15) fullfil the conditions

$$l_\mu l^{*\mu} = 1, \qquad m_\mu n^\mu = -1, \qquad 0 \quad \text{else} \tag{16}$$

Thus, from the spinorial dyad (11) a null tetrad

$$t_\mu^{(\alpha)} = (l_\mu, l^*_\mu, m_\mu, n_\mu) \tag{17}$$

can be gained such that the Minkowskian metric $\eta_{\mu\nu} = \text{diag}\ (-1, 1, 1, 1)$ turns out to be

$$\eta_{\mu\nu} = l_\mu l^*_\nu + l^*_\mu l_\nu - m_\mu n_\nu - n_\mu m_\nu \tag{18}$$

with

$$g_{(\alpha)(\beta)} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix} \tag{19}$$

With (6) we additionally find

$$m_0 = n_0 = 1, \qquad m_k = -n_k \tag{20}$$

The relations (15), (17), and (18) are based on $SL\ (2, \mathbb{C})$ as the invariance group of ur-spinors, whereas (20) is a consequence of the *SU* (2) representation according to (2), (6).

Conclusion 2. The ur-spinorial dyad (11) is *in a natural way* associated with a null-tetradial reference frame (15), (17). Relation (18) can be understood as the derivation of the pseudo-Euclidean structure of space-time from ur-tetrads.

Görnitz (1988) pointed out that the ur-theoretic cosmos $\mathbb{S}^3$ is expanding with

$$R(T) = R(0) + c \cdot T \tag{21}$$

where R is the curvature radius of $\mathbb{S}^3$, T is the cosmic epoch, and c is presumably the velocity of light. Thus, ur-theory leads to a Friedmann–Robertson–Walker cosmos with (21) in agreement with the Einstein equations.

Conclusion 3. From the ur-spinorial invariance group $SL(2, \mathbb{C}) \sim SO(1, 3)$ and from the postulate of a universal limiting velocity c we can derive the full special relativity theory from the quantum theory of urs.

Because the vectors (15) are complex, we choose real linear combinations (compare Penrose and Rindler, 1984, Vol. II, p. 120). They are explicitly given by

$$t^\mu = \frac{1}{\sqrt{2}}(m^\mu + n^\mu) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{22}$$

$$z^\mu = \frac{1}{\sqrt{2}}(m^\mu - n^\mu) = \begin{pmatrix} 0 \\ ab + a^*b^* \\ i(ab - a^*b^*) \\ bb^* - aa^* \end{pmatrix} = \begin{pmatrix} 0 \\ 2(wy - xz) \\ -1(wx + yz) \\ x^2 + y^2 - w^2 - z^2 \end{pmatrix} \tag{23}$$

$$x^\mu = \frac{1}{\sqrt{2}}(l^\mu + l^{*\mu}) = \frac{1}{2}\begin{pmatrix} 0 \\ a^2 - b^2 + a^{*2} - b^{*2} \\ i(a^2 - b^2 - a^{*2} + b^{*2}) \\ 2(ab^* + a^*b) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ x^2 - y^2 + w^2 - z^2 \\ 2(xy - wz) \\ 2(wy + xz) \end{pmatrix} \tag{24}$$

$$y^{\mu} = \frac{i}{\sqrt{2}}(l^{\mu} - l^{*\mu}) = \frac{1}{2}\begin{pmatrix} 0 \\ i(a^2 + b^2 - a^{*2} - b^{*2}) \\ -a^2 - b^2 - a^{*2} - b^{*2} \\ 2i(ab^* - a^*b) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -2(xy + wz) \\ x^2 - y^2 - w^2 + z^2 \\ 2(wx - yz) \end{pmatrix} \tag{25}$$

The system (22)–(25) is not a null tetrad. The dreibein frame (**x**, **y**, **z**) is a tangent space at the point $(w, x, y, z) = (1, 0, 0, 0)$ of $\mathbb{S}^3$ according to (5) such that (2) becomes the identity matrix $\mathbb{1}_{2\times 2}$. The dreibein (23)–(25) must be $SO(4)$-rotated in order to get a tangent space at each point of $\mathbb{S}^3$.

2.3. The Quantized Ur-Tetrad

Up to this point we have used urs as spinorial wavefunctions, i.e., we considered an ur as the first step of quantization of a simple alternative. The second quantization is done by the replacement $u_r \to \hat{a}_r$ and $u_r^* \to \hat{a}_r^+$ and the Bose commutation relations

$$[\hat{a}_r, a_s^+] = \delta_{rs}, \qquad [\hat{a}_r, \hat{a}_s] = [\hat{a}_r^+, a_s^+] = 0 \tag{26}$$

Thus, we get a quantum field theory of urs, i.e., a many-ur theory with a variable number of urs. Consequently, from (26) the quantization of the ur-tetrad (22)–(25) follows. We use a special choise of the components of the bispinorial ur $\binom{u}{u^*}$, i.e., u_r with $r = 1, \ldots, 4$ (u^* denotes an anti-ur), which belongs to a representation of $SL(2, \mathbb{C}) \oplus SL^*(2, \mathbb{C})$

$$u_1 = ae^{i\varphi}, \qquad u_2 = -b^*c^{i\varphi}, \qquad u_3 = be^{i\varphi}, \qquad u_4 = a^*\mathrm{e}^{i\varphi} \tag{27}$$
$$u_1^* = a^*e^{-i\varphi}, \qquad u_2^* = -be^{-i\varphi}, \qquad u_3^* = b^*e^{-i\varphi}, \qquad u_4^* = ae^{-i\varphi}$$

With the abbreviations

$$\hat{\tau}_{rs} = \frac{1}{2}\{\hat{a}_r^+, \hat{a}_s\}, \qquad \hat{u}_r = \hat{\tau}_{rr}, \qquad \hat{n} = \sum_r \hat{n}_r \tag{28}$$

we get

$$\hat{t}^{\mu} = \begin{pmatrix} \hat{n} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad z^{\mu} = \frac{1}{2} \begin{pmatrix} 0 \\ -\hat{\tau}_{12} - \hat{\tau}_{21} + \hat{\tau}_{34} + \hat{\tau}_{43} \\ i(\hat{\tau}_{12} - \hat{\tau}_{21} - \hat{\tau}_{34} + \hat{\tau}_{43}) \\ -\hat{n}_1 + \hat{n}_2 + \hat{n}_3 - \hat{n}_4 \end{pmatrix}$$

$$x^{\mu} = \frac{1}{2} \begin{pmatrix} 0 \\ \hat{\tau}_{14} + \hat{\tau}_{41} + \hat{\tau}_{23} + \hat{\tau}_{32} \\ i(-\hat{\tau}_{14} + \hat{\tau}_{41} - \hat{\tau}_{32} + \hat{\tau}_{23}) \\ \hat{\tau}_{13} + \hat{\tau}_{31} - \hat{\tau}_{24} - \hat{\tau}_{42} \end{pmatrix}, \tag{29}$$

$$\hat{y}^{\mu} = \frac{1}{2} \begin{pmatrix} 0 \\ i(-\hat{\tau}_{14} + \hat{\tau}_{41} - \hat{\tau}_{23} + \hat{\tau}_{32}) \\ -\hat{\tau}_{14} - \hat{\tau}_{41} + \hat{\tau}_{23} + \hat{\tau}_{32} \\ i(-\hat{\tau}_{13} + \hat{\tau}_{31} + \hat{\tau}_{24} - \hat{\tau}_{42}) \end{pmatrix}$$

Of course, this is just a first very simple version of the quantization of the global space-time model in terms of a tetradial system of ur-operators. In the language of quantum gravity $t_{\mu}^{(\alpha)}$ represent four vector bosons, i.e., massless "gravitons" with spin 1. It seems that the quantization of urs leads consequently to a quantization of the ur-tetradial reference frame, i.e., global space-time.

3. OUTLOOK

What can we learn from the quantization of the ur-tetrad about the fundamental question of whether space has to be treated as a continuum? We first look at t^{μ} in (29). The cosmic time (the epoch) is correlated with the total number of urs, i.e., the increase of the number of urs has to be understood as an expression of time. Consequently, at a certain epoch there will be only a finite number of urs in the world. This number can be estimated at about 10^{120}. Von Weizsäcker (1985, p. 471) calls this *open finitism.* If we keep the curvature radius R of $\mathbb{S}^3$ according to (21) constant, we find that with passing time there are more and more alternatives available to divide R into smaller and smaller intervalls. Equivalently, we could say that the unit sticks we use to measure spatial lengths decrease. But from open finitism, it follows that the procedure of division, i.e., "counting" of ur-alternatives, takes time, and thus space is "continuous" (i.e., infinite) only in a *potential* sense.

One hope could be that the proposed tetradial formalism is perhaps a way to deal with general relativity theory in an ur-theoretic manner. But we meet with a hard problem at this point: if we take seriously the concept of a space-time manifold which "exists" only in a potential sense, what, then, is the suitable mathematical description for it? Could it be the tetradial formalism? These questions have to be studied further.

REFERENCES

Finkelstein, D. (1994). Finite physics, in *The Universal Turing Machine—A Half-Century Survey*, R. Herken, ed., Springer, Vienna.

Görnitz Th. (1988). Connections between abstract quantum theory and space-time structure. II. A model of cosmological evolution, *International Journal of Theoretical Physics,* **27**(6):659–666.

Lyre, H. (1995). The quantum theory of ur objects as a theory of information, *International Journal of Theoretical Physics,* **34**(8):1541–1552.

Lyre, H. (1996). Multiple quantization and the concept of information, *International Journal of Theoretical Physics,* **35**(11):2219–2225.

Penrose, R., and Rindler, W. (1984). *Spinors and Space-Time I, II*, Cambridge University Press, Cambridge.

Von Weizsäcker, C. F. (1985). *Aufban der Physik*, Hanser, Munich.